International Journal of Theoretical & Applied Sciences, 10(1): 277-280(2018)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Some Fixed Point Theorems in Non-Archimedean Fuzzy Metric Spaces

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(Received 15 February 2018, Accepted 23 April, 2018)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: Some fixed point results including uniqueness are established for two mappings and triplet of mappings on Non-Archimedean Fuzzy Metric Space.
AMS Subject Classification: 47 H10, 54H25

Key Words: Fixed point, Common fixed point, continuous mapping, Non-Archimedean fuzzy metric space.

I. INTRODUCTION

After the idea of "Fuzzy Sets" which was introduced by Zadeh [23] in 1965, Deng [5], Erceg [6], Kaleva and Seikhala [13], Karamosil and Michalek [14], have introduced the concept of fuzzy metric spaces in different ways. Many authors [11], [13], [14], [20], [21] have also studied the fixed point theory in the fuzzy metric spaces and developed analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors [2], [3], [5-10], [15-16], [19].

Istrătescu and Crivăt [12] first studied Non-Archimedean probablistic metric spaces and some topological preliminaries on them. Achari [1] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istrătescu [11].

Recently Kutukcu *et. al.* [17] and Som Tammoy [22] produced significant results on Fuzzy Metric Space. Before discussing our main results we require the following definitions and lemmas as preliminaries:

II. PRELIMINARIES

Definition 2.1 [18]: A binary operation $*$: $[0,1]x[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian Topological monodies with unit 1 such that $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$ $for\ all\ a,b,c,d, \in [0,1]$

Example of t-norm are $a * b = a\,b$, $a * b = min\{a,b\}$ and $a * b = max\{a,b\}$

Definition 2.2 [14]: The 3-$tuple$ $(X,M,*)$ is called a Non-Archimedean Fuzzy metric space if X is an arbitrary set,$*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ and $s,t > 0$,

$(FM-1)$: $M(x,y,0) = 0$

$(FM-2)$: $M(x,y,t) = 1, \forall\, t \succ 0, \Leftrightarrow x = y$

$(FM-3)$: $M(x,y,t) = M(y,x,t)$

$(FM-4)$: $M(x,y,\max(t,s)) \geq M(x,y,t) * M(z,y,s)$

$(FM-5)$: $M(x,y,a) = [0,1) \to [0,1]$ is left continuous.

$(FM-6)$: if $M(x,z;u) = 1$, $M(z,y;v) = 1$ then $M(x,y;\max\{u,v\}) = 1$

$(FM-7)$: $\lim_{t\to\infty} M(x,y,t) = 1, \forall\, x,y \in X$

Note that $M\,(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y\ with\ M\,(x,y,t) = 1\ for\ all\ t > 0\ and\ M\,(x,y,t) = 0$ with ∞.

Lemma (1) [10] $for\ all\ x,y \in X$, $M(x,y,t)$ is non -decreasing.

Lemma (2) [5] Let $\{y_n\}$ be a sequence in a fuzzy metric space $(X,M,*)$ with the condition $(\boldsymbol{FM-7})$ If there exists a number $q \in (0,1)$ such that

$M(y_{n+2},y_{n+1},qt) \geq M(y_{n+1},y_n,t), \forall\, t \succ 0$ and $n = 1,2,3,\dots\dots\dots$,

Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma (3) [18] If for all $x,y \in X, t > 0$ and for a number $q \in (0,1)$,

$M(x,y,t) \geq M(xc,y,t)$, then $x = y$.

Definition 2.3: In addition of definition 2.1, if t is continuous $on\ [0,1] \times [0,1]\ and\ t(a,a) < a, a \in [0,1]$, then t is called an Archimedean t – norm. A characterization of Archimedean t – norm is due to Ling [16]. He proved that a t – norm is Archimedean if and only if it admits the representation,

$$t(a,b) = g^{(-1)}[g(a) + g(b)]$$

Where g is continuous and decreasing function form $[0,1]\ to\ [0,\infty]$ with $g(1) = 0\ and\ g(0) = \infty$ and $g^{(-1)}$ is the pseudo inverse of g.

$(go\ \ g^{(-1)})(a) = a$, for all a in the range of g.

The continuous decreasing function g appearing in this characterization. is called an additive generator of the Archimedean t-norm.

III. MAIN RESULTS

We are going to establish fixed point theorems for two and triplet of maps on complete non-Archimedean Fuzzy Metric space.

Theorem 3.1: let $(X, M, *)$ be a complete non Archimedean fuzzy metric space under the Archimedean t-norm t, with the additive generator g. Let $S\ and\ T$ be two self mappings of X in to it satisfying;

(3.1.1) $g(M(Sx,Ty,u))$
$\leq \alpha g\{M(x,y,u/\alpha) * M(x,Gx,u/\alpha) * M(y,Ty,u/\alpha) * M(y,Sx,u/\alpha)\}$
$for\ all\ x,y \in X, u > 0, and\ 0 < \alpha < 1.$

(3.1.2) $S\ and\ T$ are continuous on X.

Then $S\ and\ T$have a unique common fixed point in X.

Proof: Let $x_0 \in X$, $\{x_n\}$ be a sequence in X such that
$x_{2n+1} = Sx_{2n}, x_{2n+1} = Tx_{2n+1}, n = 0,1,2,3, \dots \dots \dots$
Be the sequence of iterates under the pair $\{S,T\}$ at x_0.
Now from (3.1.1)
$g(M(x_1,x_2,u)) = g(M(Sx_0,Tx_1,u))$
$\leq \alpha g\{M(x_1,x_2,u/\alpha)^* M(x_0,Sx_0,u/\alpha)^* M(x_1,Tx_1,u/\alpha)^* M(x_1,Sx_0,u/\alpha)\}$
$= \alpha g\{M(x_0,x_1,u/\alpha)^*\ M(x_0,x_1,u/\alpha)^* M(x_1,x_2,u/\alpha)^* M(x_1,x_1,u/\alpha)\}$
$= \alpha g(M(x_0,x_1,u/\alpha))$
$g(M(x_1,x_2,u)) \leq \alpha g(M(x_0,x_1,u/\alpha))$
Again,
$g(M(x_2,x_3,u)) = g(M(Sx_2,Tx_1,u))$
$\leq \alpha g(M(x_1,x_2,u/\alpha)) \leq \alpha^2 g(M(x_0,x_1,u/\alpha^2))$
Therefore,
$g(M(x_2,x_3,u)) \leq \alpha^2 g(M(x_0,x_1,u/\alpha^2))$
Hence it follows by induction that for every positive integer n,
$g(M(x_n,x_{n+1},u)) \leq \alpha^n g(M(x_0,x_1,u/\alpha^n))$
Now for $m > n > 0$ and $u > 0$ we have,
$M(x_{2n+1},x_{2n+2m},u)$
$\geq t\{M(x_{2n+1},x_{2n+2},u)^* M(x_{2n+2},x_{2n+2m},u)\}$
Since $\alpha < 1$ and t is non decreasing and (FM-6).
$\geq \{M(x_{2n+1},x_{2n+2},u)^* M(x_{2n+2},x_{2n+3},u)^* M(x_{2n+3},x_{2n+2m},\alpha^2 u))\}$
$= \{M(x_{2n+1},x_{2n+2},u)^* M(x_{2n+2},x_{2n+3},u)^* M(x_{2n+3},x_{2n+2m},\alpha^2 u))\}$
$= g^{-1}\{g[\{M(x_{2n+1},x_{2n+2},u)^* M(x_{2n+2},x_{2n+3},\alpha u)^*] + g[M(x_{2n+3},x_{2n+2m},\alpha^2 u)]\}$
$= g^{-1}\{g[g^{-1}\{g[M(x_{2n+1},x_{2n+2},u)] + g[M(x_{2n+2},x_{2n+3},\alpha u)] + g[M(x_{2n+3},x_{2n+2m},\alpha^2 u)]\}$
$\geq$
$g^{-1}\{g[g^{-1}\{\alpha^{2n+1} g[M(x_0,x_1,u/\alpha^{2n+1})] + \alpha^{2n+2} g[M(x_0,x_1,u/\alpha^{2n+1})] + \cdots \dots \dots \dots +$
$g[M(x_{2n+2m-1},x_{2n+2m},\alpha^{2m-2}u)]\}$
$\geq$
$g^{-1}\{g[g^{-1}\{\alpha^{2n+1} g[M(x_0,x_1,u/\alpha^{2n+1})] + \alpha^{2n+2} g[M(x_0,x_1,u/\alpha^{2n+1})] + \cdots \dots \dots \dots +$
$\alpha^{2n+2m-1} g[M(x_0,x_1,u/\alpha^{2n+1})]\}$

Hence we conclude $\{x_n\}$ be a sequence , since $g^{(-1)}$ and g are continuous,
$\alpha^n \to 0, as\ n \to \infty, FM-7\ and\ g^{(-1)}(0) = 1.$
$(X, M,*)$ is complete there is point $z \in X$ such that $x_n \to z$ the subsequences $\{x_{2n}\}, \{x_{2n+1}\}$ converges to z ie
$x_{2n} \to z,\ x_{2n+1} \to z$ continuity of $S\ and\ T$ implies that
$Sx_{2n} \to Sz,\ Tx_{2n+1} \to Tz.$
We shall now show that z is common fixed point of $S\ and\ T$ however we have,
$M(z, Sz, u) \geq t\{M(z, x_{2n}, u)M(x_{2n}, Sz, u)\}$
$= g^{(-1)}\{g[M(z, x_{2n}, u)] + g[M(x_{2n}, Sz, u)]\}$
$= g^{(-1)}\{g[M(z, x_{2n}, u)] + g[M(x_{2n-1}, Sz, u)]\}$
$= g^{(-1)}\{g[M(z, x_{2n}, u)] + \alpha g[M(x_{2n-1}, Sz, u/\alpha)]\}$
$\lim_{n\to\infty} g^{(-1)}\{g[M(z, x_{2n}, u)] + \alpha g[M(x_{2n-1}, Sz, u/\alpha)]\} = 1$
Using (3.1.1), (3.1.2) we get $Sz = z.$
Again,
$M(z, Tz, u) \geq t\{M(z, x_{2n+1}, u)M(x_{2n+1}, Tz, u)\}$
$= g^{(-1)}\{g[M(z, x_{2n+1}, u)] + g[M(x_{2n+1}, Tz, u)]\}$
$= g^{(-1)}\{g[M(z, x_{2n+1}, u)] + g[M(x_{2n}, Tz, u)]\}$
$= g^{(-1)}\{g[M(z, x_{2n+1}, u)] + \alpha g[M(x_{2n}, z, u/\alpha)]\}$
$\geq \lim_{n\to\infty} g^{(-1)}\{g[M(z, x_{2n+1}, u)] + \alpha g[M(x_{2n}, z, u/\alpha)]\} = 1$
Thus z is common fixed point of $S\ and\ T.$
In order to show that z is the only common fixed point of $S\ and\ T$, if possible let w be any other common fixed point of $S\ and\ T$
We have from (3.1.1)
$M(x, w, u) = M(Sz, Tw, u)$
$g\big(M(Sz, Tw, u)\big)$
$\leq \alpha g\{M(z, w, u/\alpha)^* M(z, Sz, u/\alpha)^* M(w, Tw, u/\alpha)^* M(w, Sz, u/\alpha)^*\}$
$= \alpha g\big(M(z, w, u/\alpha)\big)$
Therefore,
$g(M(z, w, u) \leq \alpha g(M(z, w, u/\alpha) < g\big(M(z, w, u/\alpha)\big),$ since $\alpha < 1.$
$\Longrightarrow M(z, w, u) \geq M(z, w, u/\alpha)$
Since g is decreasing function, giving a contradiction as $u/\alpha > u$
As $\alpha < 1$ and $M(x, y, u)$ is non decreasing function.Thus $z = w.$

Corollary (3.2): Let $(X, M,*)$ be a complete Non-Archimedean Fuzzy Metric Space under the Archimedean t-norm t, with the additive generator g. Let T be self mapping of X in to itself satisfying;
(3.2.1) $g\big(M(Tx, Ty, u)\big)$
$\leq \alpha g\ Max\{M(x, y, u/\alpha), M(x, Tx, u/\alpha), M(y, Ty, u/\alpha), M(x, Tx, u/\alpha)\}$
For all $x, y \in X$, $u > 0$, and $0 < \alpha < 1.$
(3.2.2) is continuous on X.
Then T has fixed point in X.

Proof: Put $S = T$, in the theorem 3.1.1
In the next theorem we further extend the results of theorem 3.1 for three self mappings.

Theorem 3.3: Let $(X, M,*)$ be a complete Non –Archimedean Fuzzy Metric Space under the $t-$norm and the additive generator g. Let $S, T\ and\ Q$ be three self mappings of X, satisfying $a * b * c * d = max(a, b, c, d)$;
(3.3.1) $g(M(SQx, TQx, u) \leq \alpha g\{M(x, y, u/\alpha)^* M(x, SQx, u/\alpha)^* M(y, TQy, u/\alpha)^* M(x, SQx, u/\alpha)^*$
For all $x, y, \in X$ $u > 0$ and $0 < \alpha < 1.$
(3.3.2) Q commutes with $S\ and\ T$, that is, $SQ = QS\ and\ TQ = QT$
(3.3.3) $S, Q\ and\ T$ are continuous on X.
Then $S, T\ and\ Q$ have a unique common fixed point in X.

Proof: Suppose $SQ = U\ and\ TQ = V$, then $U\ and\ V$ satisfy all conditions of **theorem 3.1** and therefore $U\ and\ V$ have unique common fixed point say z.

$$Uz = Vz = z\ .ie. SQz = z, TQz = z.$$

Now we shall show z is a common fixed point of $S, T\ and\ Q$.
It will be sufficient to prove $Tz = z$.
We have,
$M(z, Qz, u) = M(SQz, TQQz, u)$
Therefore from (3.3.1)
$g\big(M(SQz, TQQz, u)\big)$
$\leq \alpha g\{M(z, Qz, u/\alpha)^* M(z, SQQz, u/\alpha)^* M(Qz, TQQz, u/\alpha)^* M(Qz, SQQz, u/\alpha)\}$
$= \alpha g\{M(z, Qz, u/\alpha)^* M(z, QSQz, u/\alpha)^* M(Qz, QTQz, u/\alpha)^* M(Qz, QSQz, u/\alpha)\}$
$\alpha g\{M(z, Qz, u/\alpha)^* M(z, Qz, u/\alpha)^* M(Qz, Qz, u/\alpha)^* M(Qz, Qz, u/\alpha)\}$
$\alpha g\big(M(z, Qz, u/\alpha)\big)$
Therefore,
$g\big(M(z, Qz, u)\big) \leq \alpha g\big(M(z, Qz, u/\alpha)\big) < g\big(M(z, Qz, u/\alpha)\big)$
Since $\alpha < 1$.
$\Longrightarrow M(z, Qz, u) \geq M(z, Qz, u/\alpha)$ for all $u > 0$.
Since g is decreasing function we get a contradiction, since $u/\alpha > u\ as\ \alpha < 1$ and $M(x, y, u)$ is non decreasing function. Hence we must have $Qz = z$.
Now,
$z = Uz = SQz = Sz$
And $z = Vz = TQz = Tz$. Thus $z = Sz = Tz = Qz$.
The uniqueness of z as a common fixed point of $S, Q\ and\ T$,
Follows from the fact that z is a unique common fixed point of $SQ\ and\ TQ$.
This completes the proof.

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